STRESSES AND DISPLACEMENTS IN AN ELASTIC SHEET WEAKENED BY A DOUBLY-PERIODIC SET OF EQUAL CIRCULAR HOLES

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Koiter [1] reduced the first basic problem of the plane theory of elasticity, for an arbitrary doubly-periodic network, to a Fredholm integral equation of the second kind.

The method of solution of a doubly-periodic, plane, theory of elasticity problem, for a network formed by the exterior of congruent holes, was outlined in [2].

We present below the basis and further development of the above mentioned method. Also, the problem of reducing the doubly-periodic network to an equivalent uniform sheet is presented and solved.

1. Consider a doubly-periodic network. Let

$$\omega_1 = 2, \qquad \omega_2 = 2le^{i\alpha} \qquad (l > 0, \ \ \mathrm{Im}\,\omega_2 > 0)$$

be the basic periods, D be the region occupied by the body and λ the radius of the holes (Fig.1). Further, let L_{mn} be the contour of the hole with the center at the point $P = m\omega_1 + n\omega_2$ $(m, n = 0, \ \pm 1, \ \pm, \ldots)$, $L = UL_{mn}$ be the boundary of the region D and D° the region D with the associated boundary.

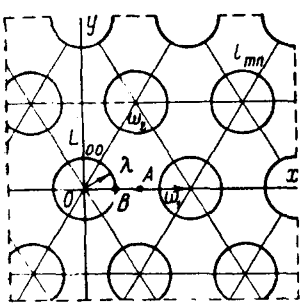

Fig. 1

For simplicity, let us assume that the network is symmetrical about the axes x and y, and that the system of forces on the contours of the holes are the same, self-balancing and possessing the same symmetry as the network. In the case of the first basic problem, the system of forces on the contours of the holes is given. In the second basic problem, the condition of its solution in the class of problems with doubly-periodic stress distributions is the quasiperiodic nature of the given displacements at the contours L_{mn}. For the symmetry of the problem, it is necessary that the displacements at the contour L_{00} be symmetrical about axes x and y.

According to [3], the above problems can be reduced to finding two functions $\Phi(z)$ and $\Psi(z)$, analytic in D, from the system of boundary conditions

$$\varepsilon\overline{\Phi(\tau)} + \Phi(\tau) - \{\bar{\tau}\Phi'(\tau) + \Psi(\tau)\} e^{2i\theta} = f_1 + if_2 \qquad (1.1)$$

$$\varepsilon = \begin{cases} 1 \\ -\varkappa \end{cases} \qquad f_1 + if_2 = \begin{cases} N - iT & \text{for the 1st basic problem} \\ -2Ge^{i\theta}d(v + iu)/ds & \text{for the 2nd basic problem} \end{cases}$$

$$\tau \in L_{mn}\ (m,n = 0, \pm 1, \pm\ , \ldots), \qquad \varkappa = (3 - \mu)/(1 + \mu)$$

Here, N and T are the normal and tangential components of the forces, acting on the contours of the holes, u and v are the displacements in the x and y directions, respectively, G is the shear modulus, μ is Poisson's ratio, θ is the angle between the normal to the contour of the hole and the x-axis and s is the coordinate distance along the curved contour L_{mn}.

The boundary conditions (1.1) on the system of contours L_{mn} can be reduced to a functional relationship on the contour of any one hole, if we subject functions $\Phi(z)$ and $\Psi(z)$ to the conditions, arising from the periodicity of the problem [2]

$$\Phi(z + \omega_1) = \Phi(z), \qquad \Phi(z + \omega_2) = \Phi(z) \qquad (1.2)$$

$$\Psi(z + \omega_1) = \Psi(z) - \bar{\omega}_1\Phi'(z), \qquad \Psi(z + \omega_2) = \Psi(z) - \bar{\omega}_2\Phi'(z)$$

The conditions of symmetry lead to Equations

$$\Phi(\bar{z}) = \overline{\Phi(z)}, \quad \Phi(-z) = \Phi(z); \quad \Psi(\bar{z}) = \overline{\Psi(z)}, \quad \Psi(-z) = \Psi(z) \quad (1.3)$$

2. Let us investigate, in the region D, a system of functions formed from the various derivatives of the functions

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{\frac{1}{(z - P)^2} - \frac{1}{P^2}\right\}, \; Q(z) = \sum_{m,n}{}' \left\{\frac{\bar{P}}{(z - P)^2} - 2z\frac{\bar{P}}{P^3} - \frac{\bar{P}}{P^2}\right\} \quad (2.1)$$

$$z = x + iy, \quad P = m\omega_1 + n\omega_2 \quad (m,n = 0 \pm 1, \pm\ , \ldots)$$

Here $\wp(z)$ is the elliptic function of Weierstrass, and $Q(z)$ is the special meromorphic function.

Our problem is to find functions $\Phi(z)$ and $\Psi(z)$, analytic in D, satisfying conditions (1.2) and (1.3). Towards the end, let us establish the necessary properties of the functions (2.1) and their derivatives.

We have the relationships

$$Q(z + \omega_1) = Q(z) + \bar{\omega}_1\wp(z) + \gamma_1$$

$$Q(z + \omega_2) = Q(z) + \bar{\omega}_2\wp(z) + \gamma_2 \qquad (2.2)$$

Indeed, substituting the function (2.1) into (2.2) and subsequently taking the derivative of (2.2), we find that the functions

$$G_1(z) = Q'(z+\omega_1) - \bar{\omega}_1 \wp'(z) - Q'(z)$$
$$G_2(z) = Q'(z+\omega_2) - \bar{\omega}_2 \wp'(z) - Q'(z) \quad (2.3)$$

are analytic in the full z-plane. We have, therefore

$$G_1(z) \equiv \text{const}, \qquad G_2(z) \equiv \text{const} \quad (2.4)$$

Let us assume in the first identity (2.4) $z = -\frac{1}{2}\omega_1$, and in the second one $z = -\frac{1}{2}\omega_2$. Considering that the function $Q'(z)$ is even and using the equations in [4]

$$\wp'(^1/_2\omega_1) = 0, \qquad \wp'(^1/_2\omega_2) = 0 \quad (2.5)$$

we obtain

$$G_1(z) \equiv 0, \qquad G_2(z) \equiv 0 \quad (2.6)$$

Integrating (2.6) leads to equations (2.2).

Let us examine the integral of the function $Q(z)$ along the contour of the parallelogram with apexes $0.5(\omega_1+\omega_2)$, $0.5(\omega_2-\omega_1)$, $-0.5(\omega_1+\omega_2)$ and $0.5(\omega_1-\omega_2)$.

Considering the regularity of $Q(z)$ in the above parallelogram and Equations (2.2), we find

$$\gamma_2\omega_1 - \gamma_1\omega_2 = \delta_1\bar{\omega}_2 - \delta_2\bar{\omega}_1, \qquad \delta_1 = 2\zeta(^1/_2\omega_1), \qquad \delta_2 = 2\zeta(^1/_2\omega_2) \quad (2.7)$$

Here $\zeta(z)$ is the zeta-function of Weierstrass. Thus, the values of γ_1 and γ_2 can be different from zero.

Differentiating (2.2) evenly, assuming in the resulting equations $z = -\frac{1}{2}\omega_1$ and $z = -\frac{1}{2}\omega_2$ and considering the fact that the function $Q^{(2k)}(z)$ is odd and $\wp^{(2k)}(z)$ $(k = 0, 1, \ldots)$, is given, we arrive at Equations

$$2Q^{(2k)}(^1/_2\,\omega_1) = \bar{\omega}_1 \wp^{(2k)}(^1/_2\,\omega_1), \qquad \gamma_1 = 2Q(^1/_2\,\omega_1) - \bar{\omega}_1\wp(^1/_2\,\omega_1) \quad (2.8)$$
$$2Q^{(2k)}(^1/_2\,\omega_2) = \bar{\omega}_2 \wp^{(2k)}(^1/_2\,\omega_2), \qquad \gamma_2 = 2Q(^1/_2\,\omega_2) - \bar{\omega}_2\wp(^1/_2\,\omega_2)$$

Note that for regular triangular $(\omega_1 = 2,\ \omega_2 = 2e^{i\pi/3})$ and square $(\omega_1 = 2,\ \omega_2 = 2i)$ networks, we have the relationship

$$\delta_1\bar{\omega}_2 - \delta_2\bar{\omega}_1 = 0 \quad (2.9)$$

In combination with the well known Legendre relationship [4]

$$\delta_1\omega_2 - \delta_2\omega_1 = 2\pi i \quad (2.10)$$

Equation (2.9) enables us to obtain in a closed form solution the network constants δ_1, δ_2, γ_1 and γ_2.

To prove Equation (2.9), let us consider a regular network and perform the transformation of the functions $\zeta(z)$, $\wp(z)$ and $Q(z)$ from the argument

z to $ze^{i\pi/3}$. By virtue of the congruency of the system $P = 2m + 2ne^{i\pi/3}$ and $Pe^{-i\pi/3} = 2n + 2me^{-i\pi/3}$ $(m, n = 0, \pm 1, \pm \ldots)$ we have the relationships

$$\zeta(ze^{i\pi/3}) = e^{-i\pi/3}\zeta(z), \quad \wp(ze^{i\pi/3}) = e^{-2i\pi/3}\wp(z), \quad Q(ze^{i\pi/3}) = e^{-3i\pi/3}Q(z) \qquad (2.11)$$

Assuming in the first equation (2.11) $z = \frac{1}{2}\omega_1$ and considering Equations $\delta_1 = 2\zeta(\frac{1}{2}\omega_1)$ and $\delta_2 = 2\zeta(\frac{1}{2}\omega_2)$, we arrive at the Equation (2.9).

Analogous considerations are also valid for a square network.

From (2.9) to (2.11) and (2.7) we can find the magnitude of the constants for regular networks.

a) Regular triangular network; $\omega_1 = 2$, $\omega_2 = 2e^{i\pi/3}$

$$\delta_1 = \frac{1}{\sqrt{3}}\pi, \qquad \delta_2 = \frac{1}{\sqrt{3}}\pi e^{-i\pi/3}, \qquad \gamma_1 = 0, \qquad \gamma_2 = 0 \qquad (2.12)$$

b) Square network; $\omega_1 = 2$, $\omega_2 = 2i$

$$\delta_1 = {}^1/_2\pi, \qquad \delta_2 = -{}^1/_2 i\pi, \qquad \gamma_2 = i\gamma_1, \qquad \gamma_1 \neq 0 \qquad (2.13)$$

3. Let us seek functions $\Phi(z)$ and $\Psi(z)$ in the form

$$\Phi(z) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2}\wp^{(2k)}(z)}{(2k+1)!}$$

$$\Psi(z) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2} \frac{\lambda^{2k+2}\wp^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2}Q^{(2k+1)}(z)}{(2k+1)!} \qquad (3.1)$$

$$\operatorname{Im} \alpha_{2k} = \operatorname{Im} \beta_{2k} = 0 \qquad (k = 0, 1, \ldots)$$

It is easy to show, using the identities (2.2), that Equations (1.2) are here satisfied.

Further, it follows from the network symmetry that the periodic systems $P = m\omega_1 + n\omega_2$ and $P^* = m^*\bar{\omega}_1 + n^*\bar{\omega}_2$ $(m, n, m^*, n^* = 0, \pm 1, \pm, \ldots)$ are congruent. Consequently, we have Equations

$$\wp^{(2k)}(z, \bar{P}) = \wp^{(2k)}(z, P), \qquad Q^{(2k+1)}(z, \bar{P}) = Q^{(2k+1)}(z, P) \qquad (3.2)$$

Thus

$$\wp^{(2k)}(\bar{z}) = \overline{\wp^{(2k)}(z)}, \qquad Q^{(2k+1)}(\bar{z}) = \overline{Q^{(2k+1)}(z)} \qquad (3.3)$$

From Equations (3.3) it immediately follows that functions (3.1) satisfy conditions (1.3). Thus, the requirements of periodicity and symmetry are satisfied. We would also note that the forms (3.1) can be obtained directly from Equations (1.2) and (1.3).

We will now impose on Expressions (3.1) the condition that the resultant vector of all the forces, acting on the curve joining two congruent points in D , be equal to zero. It is easy to see that this condition is equivalent to Equations

$$g(z+\omega_1)-g(z)=0, \qquad g(z+\omega_2)-g(z)=0 \tag{3.4}$$

where

$$g(z)=\varphi(z)+z\overline{\Phi(z)}+\overline{\psi(z)}, \qquad \varphi(z)=\int\Phi(z)\,dz, \qquad \psi(z)=\int\Psi(z)\,dz$$

Substituting Expressions (3.1) into (3.4) and considering Equations (2.2), (2.7) and (2.9), we obtain

$$\alpha_0=K_0\alpha_2\lambda^2+K_1\beta_2\lambda^2, \qquad \beta_0=K_2\alpha_2\lambda^2+K_3\beta_2\lambda^2 \tag{3.5}$$

$$K_0=\frac{\delta_1}{2\omega_1}-K_1, \qquad K_2=\frac{\bar{\gamma}_1}{\omega_1}+2K_1, \qquad K_3=2K_0, \qquad K_1=\frac{\pi i}{\bar{\omega}_1\omega_2-\omega_1\bar{\omega}_2}$$

Let us expand the functions (3.1) into Laurent series about zero

$$\Phi(z)=\sum_{k=0}^{\infty}\alpha_{2k+2}\left(\frac{\lambda}{z}\right)^{2k+2}+\sum_{k=0}^{\infty}\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}r_{j,k}z^{2j}$$

$$\Psi(z)=\sum_{k=0}^{\infty}\beta_{2k+2}\left(\frac{\lambda}{z}\right)^{2k+2}+\sum_{k=0}^{\infty}\beta_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}r_{j,k}z^{2j}- \tag{3.6}$$

$$-\sum_{k=0}^{\infty}(2k+2)\,\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}s_{j,k}z^{2j}$$

where

$$r_{j,k}=\frac{(2j+2k+1)!\,g_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}}, \qquad s_{j,k}=\frac{(2j+2k+2)!\,\rho_{j+k+1}}{(2j)!\,(2k+2)!\,2^{2j+2k+2}}$$

$$r_{0,0}=0, \qquad s_{0,0}=0$$

$$g_i=\sum_{m,n}{}'\frac{1}{T^{2i}}, \qquad \rho_i=\sum_{m,n}{}'\frac{\bar{T}}{T^{2i+1}}, \qquad T=\frac{P}{2}=m+nle^{i\alpha} \qquad (i=2,3,\ldots)$$

We shall assume, that it is possible to expand into a Fourier series the right-hand side of the functional relationship (1.1) on the contour L_{00} . Due to the symmetry of the problem, this series has the form

$$f_1+if_2=\sum_{k=-\infty}^{\infty}A_{2k}e^{2ki\theta}, \qquad \operatorname{Im}A_{2k}=0 \tag{3.7}$$

Substituting the series (3.6) and (3.7) into Equations (1.1) on L_{00} , we shall get, after some transformations, an infinite algebraic system of equations for the coefficients α_{2j+2}

$$\alpha_{2j+2}=\sum_{k=0}^{\infty}a_{j,k}\alpha_{2k+2}+b_j \qquad (j=0,1,\ldots) \tag{3.8}$$

where

$$a_{j,k} = \frac{2j+1}{\varepsilon} \gamma_{j,k} \lambda^{2j+2k+2}$$

$$\gamma_{0,0} = \frac{3g_2}{8}\lambda^2 + \varepsilon \sum_{i=1}^{\infty} \frac{(2i+1)\, g_{i+1}^2 \lambda^{4i+2}}{2^{4i+4}} + K_2 + \frac{(1+\varepsilon) K_0 K_3 \lambda^2}{1-(1+\varepsilon) K_1 \lambda^2}$$

$$\gamma_{0,k} = -\frac{(2k+2)\, \rho_{k+1}}{2^{2k+2}} + \frac{(2k+4)!\, g_{k+2}\lambda^2}{2!\,(2k+2)!\, 2^{2k+4}} + \frac{(1+\varepsilon) K_3 \lambda^2}{1-(1+\varepsilon) K_1 \lambda^2} \frac{g_{k+1}}{2^{2k+2}} +$$

$$+ \varepsilon \sum_{i=1}^{\infty} \frac{(2k+2i+1)!\, g_{i+1} g_{k+i+1} \lambda^{4i+2}}{(2k+1)!\,(2i)!\, 2^{2k+4i+4}} \qquad (k = 1, 2, \ldots)$$

$$\gamma_{j,0} = -\frac{(2j+2)\, \rho_{j+1}}{2^{2j+2}} + \frac{(2j+4)!\, g_{j+2}\lambda^2}{2!\,(2j+2)!\, 2^{2j+4}} + \frac{(1+\varepsilon) K_0 \lambda^2}{1-(1+\varepsilon) K_1 \lambda^2} \frac{g_{j+1}}{2^{2j+2}} +$$

$$+ \varepsilon \sum_{i=1}^{\infty} \frac{(2j+2i+1)!\, g_{i+1} g_{j+i+1} \lambda^{4i+2}}{(2j+1)!\,(2i)!\, 2^{2j+4i+4}} \qquad (j = 1, 2, \ldots)$$

$$\gamma_{j,k} = \gamma_{k,j} = -\frac{(2j+2k+2)!\, \rho_{j+k+1}}{(2j+1)!\,(2k+1)!\, 2^{2j+2k+2}} + \frac{(2j+2k+4)!\, g_{j+k+2}\lambda^2}{(2j+2)!\,(2k+2)!\, 2^{2j+2k+4}} +$$

$$+ \varepsilon \sum_{i=0}^{\infty} \frac{(2j+2i+1)!\,(2k+2i+1)!\, g_{j+i+1} g_{k+i+1} \lambda^{4i+2}}{(2j+1)!\,(2k+1)!\,(2i+1)!\,(2i)!\, 2^{2j+2k+4i+4}} +$$

$$+ \frac{g_{j+1} g_{k+1} \lambda^2}{2^{2j+2k+4}} \left\{1 + \frac{(1+\varepsilon)^2 K_1 \lambda^2}{1-(1+\varepsilon) K_1 \lambda^2}\right\} \qquad (j,\, k = 1, 2, \ldots)$$

$$\varepsilon b_0 = A_2 - \frac{A_0 K_3 \lambda^2}{1-(1+\varepsilon) K_1 \lambda^2} = \sum_{k=0}^{\infty} \frac{g_{k+2} \lambda^{2k+4}}{2^{2k+4}} A_{-2k-2}$$

$$\varepsilon b_j = A_{2j+2} - \frac{(2j+1) A_0 \lambda^{2j+2}}{1-(1+\varepsilon) K_1 \lambda^2} \frac{g_{j+1}}{2^{2j+2}} - \sum_{k=0}^{\infty} \frac{(2j+2k+3)!\, g_{j+k+2} \lambda^{2k+2j+4}}{(2j)!\,(2k+3)!\, 2^{2j+2k+4}} A_{-2k-2}$$

For the constants β_{2j+2} we get Equations (3.9)

$$\beta_2 = \frac{1}{1-(1+\varepsilon) K_1 \lambda^2} \left\{- A_0 + (1+\varepsilon) K_0 \lambda^2 \alpha_2 + (1+\varepsilon) \sum_{k=1}^{\infty} \frac{g_{k+1} \lambda^{2k+2}}{2^{2k+2}} \alpha_{2k+2}\right\}$$

$$\beta_{2j+4} = (2j+3)\, \alpha_{2j+2} + \varepsilon \sum_{k=0}^{\infty} \frac{(2j+2k+3)!\, g_{j+k+2} \lambda^{2j+2k+4}}{(2j+2)!\,(2k+1)!\, 2^{2j+2k+4}} \alpha_{2k+2} - A_{-2j-2} \; (j=0,1,\ldots)$$

This concludes the construction of the solution.

4. We shall show, that if the holes in the network do not touch or intersect, then, with certain assumptions on the loading, the forms (3.1), together with Equations (3.8) and (3.9), give required solution.

It is easy to derive the following valid estimates

$$|g_k| < \frac{M}{\lambda_1^{2k}}, \qquad |\rho_k| < \frac{M}{\lambda_1^{2k}}, \qquad M > 0 \qquad (k = 2, 3, \ldots) \tag{4.1}$$

where M is some number, and λ_1 is one half of the distance between two closest congruent points.

Let the coefficients A_{2k}, in the expansion (3.7), satisfy the inequality

$$|A_j| < \frac{A}{j^{h+1}}, \qquad A > 0, \qquad h > 0 \tag{4.2}$$

Under the conditions (4.2), the system (3.8) has a unique finite solution, with $\lim \alpha_{2j+2} = 0$ when $j \to \infty$.

Indeed, using (4.1), it is easy to get estimates (*) for the coefficients and free terms of the system (3.8)

$$|a_{j,k}| \leqslant M\,(2j+1)\,\delta^{2j+2k+2}, \qquad |b_j| \leqslant M\left\{\delta^{2j} + \left(\frac{1}{2j+2}\right)^{h+1}\right\}, \qquad 0 \leqslant \delta = \frac{\lambda}{\lambda_1} < 1 \tag{4.3}$$

As a result of the first estimate (4.3), the double sum $\Sigma|a_{j,k}|$ converges, which, together with the second estimate (4.3) provides a sufficient condition for the existance of a unique finite solution of the system (3.8).

Using the estimates (4.3), we get from (3.8)

$$|\alpha_{2j+2}| \leqslant M\left\{(2j+1)\,\delta^{2j} + \left(\frac{1}{2j+2}\right)^{h+1}\right\}, \qquad 0 \leqslant \delta < 1 \qquad (j = 0, 1 \ldots) \tag{4.4}$$

The assertion has thus been proved.

For values of β_{2j+2} we have estimates

$$|\beta_{2j+2}| \leqslant M\left\{(4j^2-1)\,\delta^{2j-2} + \left(\frac{1}{2j+2}\right)^{h}\right\}, \qquad 0 \leqslant \delta < 1 \qquad (j = 0, 1, \ldots) \tag{4.5}$$

Let us investigate the functions

$$\varphi(z) = \alpha_0 z - \alpha_2\lambda^2\zeta(z) + \sum_{k=1}^{\infty} \alpha_{2k+2}\frac{\lambda^{2k+2}\wp^{(2k-1)}(z)}{(2k+1)!}$$

$$\psi(z) = \beta_0 z - \beta_2\lambda^2\zeta(z) + \sum_{k=1}^{\infty} \beta_{2k+2}\frac{\lambda^{2k+2}\wp^{(2k-1)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2}\frac{\lambda^{2k+2}Q^{(2k)}(z)}{(2k+1)!} \tag{4.6}$$

$$\varphi(z) = \int \Phi(z)\,dz, \qquad \psi(z) = \int \Psi(z)\,dz$$

With $0 \leqslant \delta \leqslant r < 1$, the series (4.6) converge absolutely and uniformly in the closed region D°.

*) Under condition $0 \leqslant \lambda < \lambda_1$, Expression $1 - (1+\varepsilon)K_1\lambda^2$ is non zero.

Indeed, it is easy to show, that if z is located in the part of D°, inside the basic periodic parallelogram, we have the inequalities

$$\sum_{m,n}\left|\frac{1}{(z-P)^{k+2}}\right| < \frac{M}{\lambda^{k+2}} \qquad (k=1,2,\ldots) \tag{4.7}$$

Using (2.1), we have the estimates in D°

$$\left|\frac{\wp^{(k)}(z)}{(k+1)!}\right| < \frac{M}{\lambda^{k+2}}, \qquad \left|\frac{Q^{(k)}(z)}{(k+1)!}\right| < \frac{M}{\lambda^{k+2}} \qquad (k=0,1,\ldots) \tag{4.8}$$

Using (4.4), (4.5) and (4.8), we get

$$\begin{aligned} \sum_{k=1}^{\infty}\left|\alpha_{2k+2}\frac{\lambda^{2k+2}\wp^{(2k-1)}(z)}{(2k+1)!}\right| &< A\sum_{k=1}^{\infty}\left(\delta^{2k} + \frac{1}{k^{h+2}}\right) \\ \sum_{k=1}^{\infty}\left|\beta_{2k+2}\frac{\lambda^{2k+2}\wp^{(2k-1)}(z)}{(2k+1)!}\right| &< A\sum_{k=1}^{\infty}\left(k\delta^{2k-2} + \frac{1}{k^{h+1}}\right) \\ \sum_{k=1}^{\infty}\left|\alpha_{2k+2}\frac{\lambda^{2k+2}Q^{(2k)}(z)}{(2k+1)!}\right| &< A\sum_{k=1}^{\infty}\left(k\delta^{2k} + \frac{1}{k^{h+1}}\right) \end{aligned} \tag{4.9}$$

The assertion has thus been proved. The regularity of the functions $\varphi(z)$ and $\psi(z)$ in D and their continuity in D° follow from the relationship

$$\begin{aligned} \varphi(z+\omega_1) &= \varphi(z) + C_1, \qquad \psi(z+\omega_1) = \psi(z) - \bar{\omega}_1\Phi(z) + a_1 \\ \varphi(z+\omega_2) &= \varphi(z) + C_2, \qquad \psi(z+\omega_2) = \psi(z) - \bar{\omega}_2\Phi(z) + a_2 \end{aligned} \tag{4.10}$$

which are satisfied by the series (4.6).

The basic problems for the doubly-periodic network, formulated in Section 1, could have been presented as boundary problems for the functions $\varphi(z)$ and $\psi(z)$. Also, the system (3.8) and Equations (3.9) could have been obtained starting from the representation (4.6). All the operations performed on (4.6) to obtain (3.8) and (3.9) are justified because of the absolute convergence (4.6) in the closed region D°.

Thus, if the constants α_{2k+2} satidfy the system (3.8), and the constants β_{2k+2} are obtained from (3.9), then, with $0 \leqslant \lambda < \lambda_1$, the series (4.6) define two functions $\varphi(z)$ and $\psi(z)$ which are analytic in D and continuous in D° and represent the solution of the stated problem.

5. Let us investigate the problem of reducing the doubly-periodic network to an equivalent uniform sheet.

Let there be in the network the average stresses $\sigma_x^\circ = \sigma_1$, $\sigma_y^\circ = \sigma_2$ and $\tau_{xy}^\circ = \tau_{12} = 0$. The functions $\Phi(z)$ and $\Psi(z)$ have here the form

$$\Phi_s(z) = {}^1/_4(\sigma_1+\sigma_2) + \Phi(z), \qquad \Psi_s(z) = {}^1/_2(\sigma_2-\sigma_1) + \Psi(z) \tag{5.1}$$

where $\Phi(z)$ and $\Psi(z)$ are defined by the series (3.1).

It is easy to see that the displacements, in the class of problems defined by the functions (5.1), are quasiperiodic functions.

Indeed, the displacements have the form [3]

$$2G\,(u+iv)=h_s(z)=\varkappa\varphi_s(z)-z\overline{\Phi_s(z)}-\overline{\psi_s(z)}$$
$$\varphi_s(z)=\int\Phi_s(z)\,dz,\qquad \psi_s(z)=\int\Psi_s(z)\,dz \tag{5.2}$$

Let us assume

$$\Omega_1=h_s(z+\omega_1)-h_s(z),\qquad \Omega_2=h_s(z+\omega_2)-h_s(z) \tag{5.3}$$

Substituting the series (4.6) into (5.3), and using (5.1), (2.2) and (2.8) we get

$$\Omega_1=\frac{\sigma_1+\sigma_2}{2}\frac{1-\mu}{1+\mu}\omega_1-\frac{\sigma_2-\sigma_1}{2}\bar{\omega}_1+\alpha_2\lambda^2(\bar{\gamma}_1-\varkappa\delta_1)+\beta_2\lambda^2\bar{\delta}_1+$$
$$+(\varkappa-1)\,\alpha_0\omega_1-\beta_0\bar{\omega}_1 \tag{5.4}$$
$$\Omega_2=\frac{\sigma_1+\sigma_2}{2}\frac{1-\mu}{1+\mu}\omega_2-\frac{\sigma_2-\sigma_1}{2}\bar{\omega}_2+\alpha_2\lambda^2(\bar{\gamma}_2-\varkappa\delta_2)+\beta_2\lambda^2\bar{\delta}_2+(\varkappa-1)\,\alpha_0\omega_2-\beta_0\bar{\omega}_2$$

Let us now consider an orthotropic uniform sheet under uniform tension $\sigma_x=\sigma_1$, $\sigma_y=\sigma_2$, $\tau_{xy}=0$.

Hook's law, for an orthotropic medium, has the form [5]

$$\varepsilon_x=\frac{\sigma_x}{E_1^*}-\frac{\mu_2^*\sigma_y}{E_2^*},\quad \varepsilon_y=\frac{\sigma_y}{E_2^*}-\frac{\mu_1^*\sigma_x}{E_1^*},\quad \gamma_{xy}=\frac{\tau_{xy}}{G^*},\quad E_1^*\mu_2^*=E_2^*\mu_1^* \tag{5.5}$$

Here E_1^* and E_2^* are the moduli of elasticity in the direction of the main axes x and y, μ_1^* and μ_2^* are the corresponding Poisson's ratios and G^* the modulus of the elastisity of the second type.

From (5.5), we can find the displacements of points in the uniform sheet

$$\frac{h^*(z)}{2G}=u+iv=\frac{z}{2}\left(\sigma_1\frac{1-\mu_1^*}{E_1^*}+\sigma_2\frac{1-\mu_2^*}{E_2^*}\right)+\frac{\bar{z}}{2}\left(\sigma_1\frac{1+\mu_1^*}{E_1^*}-\sigma_2\frac{1+\mu_2^*}{E_2^*}\right) \tag{5.6}$$

From here we have

$$\Omega_1^*=h^*(z+\omega_1)-h^*(z)=G\omega_1\left(\sigma_1\frac{1-\mu_1^*}{E^*}+\sigma_2\frac{1-\mu_2^*}{E_2^*}\right)+$$
$$+G\bar{\omega}_1\left(\sigma_1\frac{1+\mu_1^*}{E_1^*}-\sigma_2\frac{1+\mu_2^*}{E_2^*}\right)$$
$$\Omega_2^*=h^*(z+\omega_2)-h^*(z)=G\omega_2\left(\sigma_1\frac{1-\mu_1^*}{E_1^*}+\sigma_2\frac{1-\mu_2^*}{E_2^*}\right)+$$
$$+G\bar{\omega}_2\left(\sigma_1\frac{1+\mu_1^*}{E_1^*}-\sigma_2\frac{1+\mu_2^*}{E_2^*}\right) \tag{5.7}$$

Equations (5.7) establish the quasiperiodicity of the displacements (5.6).

We shall now introduce the equivalence notion between the uniform sheet and the doubly-periodic network. In general, an equivalent uniform sheet is understood to be a sheet, the stiffness of which is equal to the stiffness of the sheet with the holes. This means, that under the same loading the average displacements of such sheets are equal. For the class of problems under investigation, these displacements are quasiperiodic functions.

We shall denote the doubly-periodic network and the uniform sheet as equivalent, if under the same loading we have equalities

$$\Omega_1 = \Omega_1^*, \qquad \Omega_2 = \Omega_2^* \tag{5.8}$$

where Ω_1 and Ω_2 are defined by (5.4), and Ω_1^* and Ω_2^* by Equations (5.7).

Equations (5.8) reflect the fact that the displacements of any point z relative to its congruent point $z + m\omega_1 + n\omega_2$ ($m, n = 0, \pm 1, \pm, \ldots$) in the uniform sheet and in the doubly-periodic one are equal.

Substituting Equations (5.4) and (5.7) into (5.8), we get

$$\begin{aligned}
&\frac{\sigma_1+\sigma_2}{2}\frac{1-\mu}{E}\omega_1 - \frac{\sigma_2-\sigma_1}{2}\frac{1+\mu}{E}\bar{\omega}_1 + \frac{1+\mu}{E}\{\alpha_2\lambda^2(\bar{\gamma}_1 - \varkappa\delta_1) + \beta_2\lambda^2\bar{\delta}_1 + \\
&\quad + (\varkappa-1)\alpha_0\omega_1 - \beta_0\bar{\omega}_1\} = \frac{\omega_1}{2}\left(\sigma_1\frac{1-\mu_1^*}{E_1^*} + \sigma_2\frac{1-\mu_2^*}{E_2^*}\right) + \\
&\quad + \frac{\bar{\omega}_1}{2}\left(\sigma_1\frac{1+\mu_1^*}{E_1^*} - \sigma_2\frac{1+\mu_2^*}{E_2^*}\right) \\
&\frac{\sigma_1+\sigma_2}{2}\frac{1-\mu}{E}\omega_2 - \frac{\sigma_2-\sigma_1}{2}\frac{1+\mu}{E}\bar{\omega}_2 + \frac{1+\mu}{E}\{\alpha_2\lambda^2(\bar{\gamma}_2 - \varkappa\delta_2) + \beta_2\lambda^2\bar{\delta}_2 + \\
&\quad + (\varkappa-1)\alpha_0\omega_2 - \beta_0\bar{\omega}_2\} = \frac{\omega_2}{2}\left(\sigma_1\frac{1-\mu_1^*}{E_1^*} + \sigma_2\frac{1-\mu_2^*}{E_2^*}\right) + \\
&\quad + \frac{\bar{\omega}_2}{2}\left(\sigma_1\frac{1+\mu_1^*}{E_1^*} - \sigma_2\frac{1+\mu_2^*}{E_2^*}\right)
\end{aligned} \tag{5.9}$$

Expressions (5.9) provide a complete system of equations for the determination of the three independent reduced elastic parameters E_1^*, E_2^* and μ_1^*. For the determination of the fourth parameter G^* we can form analogous expressions for average stresses $\sigma_1 = \sigma_2 = 0$, $\tau_{12} = \tau$.

We shall form them for the case of a square network.

Let us first assume in (5.9) that $\sigma_1 = \sigma_2 = \sigma$. Then, the first equation (5.9) yields

$$\frac{E_1^*/(1-\mu_1^*)}{E/(1-\mu)} = \left\{1 + \frac{1}{1-\mu}\,[4K_1\beta_2\lambda^2 - (\delta_1 + 4K_1)\,\alpha_2\lambda^2]\right\}^{-1} \tag{5.10}$$

Here α_2 and β_2 must be obtained from the solution of the corresponding doubly-periodic problem with average stresses $\sigma_1 = \sigma_2 = \sigma$, $\tau_{12} = 0$.

Further, multiplying the first expression (5.9) by ω_2, the second by ω_1, subtracting the second expression from the first and using (2.7) and (2.10) we obtain

$$\frac{E}{E_2^*} - \frac{E}{E_1^*} = \alpha_2\lambda^2\left\{\frac{16\pi i}{\bar{\omega}_1\omega_2 - \omega_1\bar{\omega}_2} + 2\,(1+\mu)\frac{\bar{\delta}_1\omega_2 - \bar{\delta}_2\omega_1}{\bar{\omega}_1\omega_2 - \omega_1\bar{\omega}_2}\right\} \tag{5.11}$$

In Equation (5.11), the value of α_2 is the same as in Equation (5.10).

Let now the average stresses be $\sigma_1 = -\sigma_2 = \sigma$, $\tau_{12} = 0$.

From the first equation (5.9), we obtain

$$\frac{E_1^*/(1+\mu_1^*)}{E/(1+\mu)} = \left[1 + 4K_1\beta_2\lambda^2 - \frac{\lambda^2\alpha_2}{1+\mu}(\delta_1 + 4K_1)\right]^{-1} \tag{5.12}$$

Here, the value of α_2 and β_2 must be determined from the solution of the doubly-periodic problem with average stresses $\sigma_1 = -\sigma_2 = \sigma$, $\tau_{12} = 0$.

Combining the first and second expressions (5.9), we obtain an equation equivalent to (5.11).

Let us investigate the case, more interesting from a practical point of view, of a regular network. In this case, in solving the doubly-periodic problem with average stresses $\sigma_1 = \sigma_2 = \sigma$, $\tau_{12} = 0$ the magnitude of $\alpha_2 = 0$ and β_2 is non zero. Consequently, from (5.11) we obtain

$$E_1^* = E_2^* = E^* \tag{5.13}$$

Further, from (3.5) we find

$$K_1 = \frac{\pi}{8\sin\alpha} = \frac{\delta_1}{4} \qquad \left(\alpha = \frac{\pi}{3}, \frac{\pi}{2}\right) \tag{5.14}$$

Substituting the value of K_1 from (5.14) into Equations (5.10) and (5.12) and considering the fact that in the solution of the problem with average stresses $\sigma_1 = -\sigma_2 = \sigma$, $\tau_{12} = 0$, we have $\beta_2 = 0$ and $\alpha_2 \neq 0$, we obtain

a) Regular triangular network; $\omega_1 = 2$, $\omega_2 = 2e^{i\pi/3}$

$$\frac{E^* | (1-\mu^*)}{E | (1-\mu)} = \left[1 + \frac{\pi}{\sqrt{3}}\frac{\beta_2\lambda^2}{1-\mu}\right]^{-1}, \qquad \frac{E^* | (1+\mu^*)}{E | (1+\mu)} = \left[1 - \frac{2\pi}{\sqrt{3}}\frac{\alpha_2\lambda^2}{1+\mu}\right]^{-1} \tag{5.15}$$

b) Square network; $\omega_1 = 2$, $\omega_2 = 2i$

$$\frac{E^* | (1-\mu^*)}{E | (1-\mu)} = \left[1 + \frac{\pi}{2}\frac{\beta_2\lambda^2}{1-\mu}\right]^{-1}, \qquad \frac{E^* | (1+\mu^*)}{E | (1+\mu)} = \left[1 - \pi\frac{\alpha_2\lambda^2}{1+\mu}\right]^{-1} \tag{5.16}$$

Since the regular triangular network is isotropic in the sense of the reduced elastic parameters, it follows that Equations (5.15) determine fully the elastic characteristics of the equivalent uniform sheet.

For the square network it is necessary to determine the magnitude of G^*. For this reason, let us investigate a doubly-periodic network with average stresses $\sigma_1 = \sigma_2 = 0$, $\tau_{12} = \tau$.

Performing an anticlockwise rotation of the coordinate system of 45°, we change the problem to a symmertical one with average stresses, in the new coordinate system $x'oy'$: $\sigma_{x'}^{\circ} = -\sigma_{y'}^{\circ} = \tau$, $\tau_{x'y'} = 0$.

Following the same steps as in the derivation of Equations (5.9), we obtain

$$\begin{aligned}\bar{\omega}_1' + \alpha_2\lambda^2(\bar{\gamma}_1' - \varkappa\delta_1') - \alpha_2\lambda^2(\delta_1 + \bar{\gamma}_1') &= \frac{E}{1+\mu}\frac{\bar{\omega}_1'}{2G^*} \\ \bar{\omega}_2' + \alpha_2\lambda^2(\bar{\gamma}_2' - \varkappa\delta_2') - \alpha_2\lambda^2(\delta_2 + \bar{\gamma}_2') &= \frac{E}{1+\mu}\frac{\bar{\omega}_2'}{2G^*}\end{aligned} \tag{5.17}$$

where the primed quantities refer to the basic periods $\omega_1' = 2e^{-i\pi/4}$ and $\omega_2' = 2e^{i\pi/4}$ of the network in the system of coordinates $x'oy'$.

Using the following easily established equalities

$$\delta_1' = \delta_1 e^{i\pi/4}, \quad \delta_2' = \delta_2 e^{i\pi/4}, \quad \gamma_1' = \gamma_1 e^{3i\pi/4}, \quad \gamma_2' = \gamma_2 e^{3i\pi/4} \tag{5.18}$$

we obtain from (5.17)

$$\frac{G^*}{G} = \left[1 - \pi\frac{\alpha_2\lambda^2}{1+\mu}\right]^{-1} \tag{5.19}$$

The condition of compatibility of Equations (5.17) is the relationship (2.9) which is satisfied for regular networks.

We present the results of the calculations of stresses $\sigma_\theta(A)$ and $\sigma_\theta(B)$ at representative points A and B. To check the boundary conditions, we also show the values of the stress $\sigma_r(B)$ at point B.

1. Regular triangular network; $\omega_1 = 2$, $\omega_2 = 2e^{i\pi/3}$; the edges of the holes are force free

a) Average stresses $\sigma_1 = 1$, $\sigma_2 = 1$, $\tau_{12} = 0$

λ	0.2	0.4	0.6	0.8	0.9
$\sigma_\theta(A)$	1.10	1.45	2.27	4.70	9.66
$\sigma_\theta(B)$	2.07	2.35	3.09	5.70	10.70
$\sigma_r(B)$	0.0000	0.0000	0.0000	0.0004	0.0087

b) Average stresses $\sigma_2 = 1$, $\sigma_1 = 0$, $\tau_{12} = 0$

λ	0.2	0.4	0.6	0.8	0.9
$\sigma_\theta(A)$	1.03	1.23	2.00	4.72	9.20
$\sigma_\theta(B)$	3.09	3.31	3.62	5.55	10.70
$\sigma_r(B)$	0.0000	−0.0000	0.0018	− 0.008	0.060

2. Regular triangular network, the holes contain rigid rings

a) Average stresses $\sigma_1 = 1$, $\sigma_2 = 1$, $\tau_{12} = 0$

λ	0.2	0.4	0.6	0.8	0.9
$\sigma_r(A)$	1.01	1.04	1.10	1.20	1.47
$\sigma_r(B)$	1.51	1.43	1.32	1.27	1.55
$\sigma_\theta(A)$	0.95	0.81	0.61	0.38	0.26
$\sigma_\theta(B)$	0.45	0.43	0.40	0.38	0.26

b) Average stresses $\sigma_1 = 1$, $\sigma_2 = 0$, $\tau_{12} = 0$

λ	0.2	0.4	0.6	0.8	0.9
$r(A)$	1.06	1.23	1.43	1.64	1.90
$\sigma_r(B)$	1.49	1.44	1.44	1.59	1.90
$\sigma_\theta(A)$	−0.03	− 0.09	−0.08	0.12	0.32
$\sigma_\theta(B)$	0.44	0.43	0.43	0.46	0.46

3. Square network; $\omega_1 = 2$, $\omega_2 = 2i$; the edges of the holes are force free

a) Average stresses $\sigma_1 = 1$, $\sigma_2 = 1$, $\tau_{12} = 0$

$\lambda =$	0.2	0.4	0.6	0.8	0.9
$\sigma_\theta(A) =$	1.10	1.46	2.25	4.65	9.58
$\sigma_\theta(B) =$	2.07	2.36	3.19	5.75	10.63
$\sigma_r(B) =$	0.0000	0.0001	0.0001	—0.0050	—0.0289

b) Average stresses $\sigma_2 = 1$, $\sigma_1 = 0$, $\tau_{12} = 0$

$\lambda =$	0.2	0.4	0.6	0.8	0.9
$\sigma_\theta(A) =$	1.15	1.34	2.10	4.63	9.51
$\sigma_\theta(B) =$	3.00	3.11	3.61	5.78	10.60
$\sigma_r(B) =$	0.0000	—0.0002	—0.0009	—0.0017	—0.0060

4. Square network and the holes contain rigid rings

a) Average stresses $\sigma_1 = 1$, $\sigma_2 = 1$, $\tau_{12} = 0$

$\lambda =$	0.2	0.4	0.6	0.8	0.9
$\sigma_r(A) =$	1.02	1.08	1.18	1.44	1.86
$\sigma_r(B) =$	1.51	1.45	1.39	1.48	1.83
$\sigma_\theta(A) =$	0.95	0.80	0.58	0.35	0.41
$\sigma_\theta(B) =$	0.45	0.44	0.42	0.45	0.56

b) Average stresses $\sigma_1 = 1$, $\sigma_2 = 0$, $\tau_{12} = 0$

$\lambda =$	0.2	0.4	0.6	0.8	0.9
$\sigma_r(A) =$	1.07	1.25	1.48	1.74	2.12
$\sigma_r(B) =$	1.50	1.50	1.52	1.71	2.08
$\sigma_\theta(A) =$	0.02	—0.04	0.00	0.20	0.43
$\sigma_\theta(B) =$	0.45	0.45	0.46	0.51	0.65

Using the solutions to the problem given above, we can establish the reduced elastic parameters of regular networks

The curves of E^*/E and μ^*/μ for regular triangular networks, when the edges of the holes are force free, are given in Fig.2.

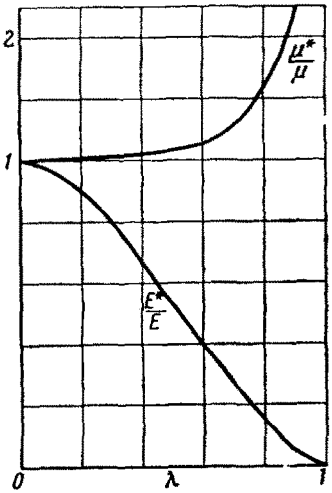

Fig. 2

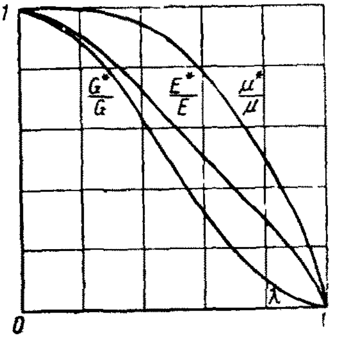

Fig. 3

For the case when the holes contain rigid rings, the respective values are given in Fig.4.

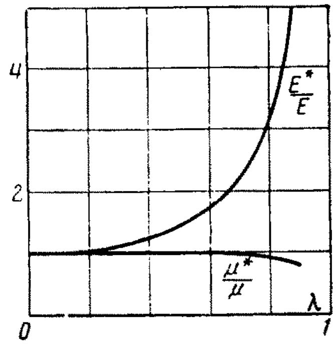

Fig. 4

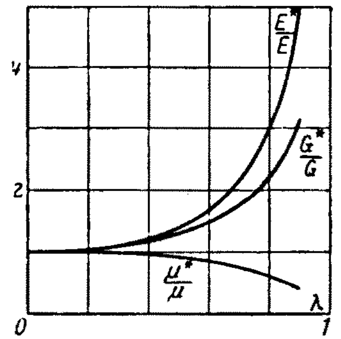

Fig. 5

The curves of E^*/E, μ^*/μ and G^*/G for a square network, when the edges of the holes are force free, are given in Fig.3.

The variations of the same variables for the case when the holes contain rigid rings is shown in Fig.5.

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